The Resonant Frequency and Tuning Characteristics of a Narrow-Gap Reentrant Cylindrical Cavity

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Abstract—**The literature concerning the reentrant cylindrical cavity is reviewed and the relative advantages and disadvantages of the various formulations of the problem are discussed. In addition, a new formulation is proposed which accurately predicts the resonant frequency of narrow-gap cavities such as those currently finding application in the construction of solid-state oscillators. This new formulation is mathematically simpler and numerically more efficient than many other formulations which are not as accurate. The paper concludes with an investigation of the tuning characteristics of the cavity.**

I. Introduction

THE reentrant cylindrical cavity shown in Fig. 1 was first investigated almost 40 years ago in connection with the development of klystrons, and since that time a number of interesting papers have appeared on the subject all attempting to further our understanding of this basic but most useful cavity. The simple mechanical construction and wide tuning range are characteristics of this cavity which can be usefully employed in the design of microwave oscillators, and it is therefore not surprising that along with the recent developments in solid-state devices, and in particular the advent of tunnel and Gunn diodes, there has come renewed interest in reentrant cylindrical cavities. Indeed, a few papers have appeared recently in which this cavity has been investigated for geometries appropriate to such semiconductor devices [2], [3].

The purpose of this paper is twofold: firstly, to attempt a reasonably complete review of the literature on the subject, and secondly, to propose a reasonably accurate, numerically simple method of calculating the fundamental resonant frequency of a narrow-gap cavity.

II. Review

The first analysis of the reentrant cylindrical cavity was presented by Hansen [1] who considered the structure to be composed of the two regions, I and II, as shown in Fig. 1, having the surface $r = a$, $0 \leq z \leq g$ in common. By approximating the electric field on this surface and matching the magnetic field results obtained for the two regions at $r = a$ and $z = g$, Hansen obtained a transcendental equation from which the resonant frequency could be calculated. (NB, Hansen's approximation for the electric field, which was intended to be a good approximation when g was small, is, in fact, considerably in error [4], [5].) The resonant frequency predicted by Hansen's analysis was generally within a few percent of the correct result and thus

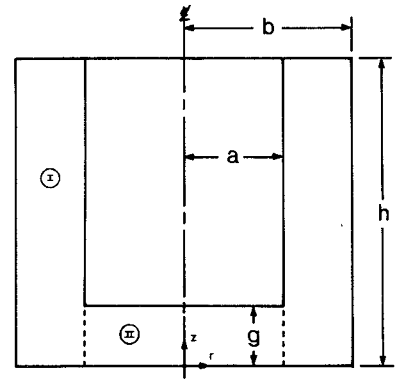

Fig. 1. Cross section of the reentrant cylindrical cavity.

sufficiently accurate for many applications, and for this reason results computed from his formulation have had widespread publication [6]–[8].

In 1941 Hahn [9] proposed an alternative formulation in which the field quantities were first expanded as finite Fourier series after which the boundary conditions were applied and the unknown coefficients evaluated. While Hahn's work has been used and referenced by other authors it appears to have had relatively little application compared to Hansen's.

By 1946 it had been observed that Hansen's formulation became inaccurate when $kh > \pi/2$, and in an effort to overcome this deficiency Mayer [10] proposed a solution based on the variational method of Schwinger. Using his formulation Mayer predicted the resonant frequencies of a number of long reentrant cavities with an accuracy of about 2 percent. It must be pointed out, however, that the accuracy obtained by Mayer was achieved at some expense in that his variationally derived expression was very complex both mathematically and numerically.

Subsequently, there was interest in obtaining a mathematically simple engineering solution for the problem, and to this end an approximate formulation was proposed by Kihara [11], while a decade later a general treatment of klystron cavities was given by Fujisawa [12]. (A more mathematically detailed consideration of general cavities was earlier given by Hansen and Richtmyer [13].) In his analysis Kihara made gross approximations to the magnetic field in order to simplify the form of the solution, and, as a consequence, results predicted by his formulation only qualitatively illustrate the electrical behavior of the cavity. Fujisawa, on the other hand, used a Green's function approach to propose an equivalent circuit for a general

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cylindrical cavity. This equivalent circuit predicts the resonant frequency of a reentrant cylindrical cavity to within a few percent provided the cavity is neither too flat nor, with a thin center post, too long.

Theoretical and experimental studies have also been made of the reentrant cylindrical cavity with an offset gap [15], [16], and it is interesting to note that in addition to the previously mentioned analytical studies there has also been the occasional study using some analogous static situation [14].

Some of the approximate analyses mentioned previously, together with those which may be obtained by considering the cavity as a coaxial line region supporting only the TEM mode and so on, are, in many cases, not sufficiently accurate for design purposes [2], [3], especially for many of the problems of current interest. As a result there has recently been some reinvestigation of the reentrant cylindrical cavity. In particular Uenakada [2] used a combined Green's function and variational approach to compute the resonant frequencies of a set of cavities whose dimensions were typical of those used in the construction of solid-state equipment. Like Hansen, Uenakada chose the surface $r = a$, $0 \le z \le g$ to be the division between the two cylindrical regions and he imposed the condition that at resonance the total admittance (i.e., the sum of the admittances of regions I and II) at this boundary must be zero. He calculated the admittance of region I using the Green's function together with a variational expression for the admittance in which he used the trial function

$$M = 1 + m\cos\frac{\pi z}{g}.$$

This function was proportional to $E_z(a,z)$, the electric field in the gap at $r = a$, $0 \le z \le g$, and Uenakada treated m as the variational variable which he determined by imposing the condition $dY/dm = 0$, where Y was the admittance of region I as seen at the gap. His final expression for the admittance Y was quite complicated and required the summation of three infinite series, the numerical evaluation of which does not appear especially easy. (The details of the numerical evaluation procedure were not given by Uenakada.) Furthermore, in every term of all three infinite series there occurs a common integral which must be evaluated numerically. Finally, he assumed that the admittance seen looking into the radial transmission line region (i.e., region II) was that due to a TEM mode alone. In an experimental study Uenakada demonstrated his theoretical approach to be accurate to about 3 or 4 percent (on the average) for cavities with narrow gaps (see Table I).

In their paper Rivier and Vergé-Lapisardi [3] pointed out that a simple LC equivalent circuit may be used to approximately represent the cavity where the values of the components of the equivalent circuit may be deduced from experimental measurements. They confirmed this by demonstrating the approximately linear region of the $(1/40)\cdot(\lambda_{res}/2a)^2(g/h)$ versus (g/h) curve.

It was this author's contention that it was possible to evaluate, with reasonable accuracy, the properties, particularly the resonant frequency, of a reentrant cylindrical cavity satisfying (1) by means which do not involve the mathematical and computational complexities found in many previous formulations. Such a theory was developed as will be outlined.

III. Formulation of the Problem

Consider now the reentrant cylindrical cavity shown in cross section in Fig. 1 with particular interest in cases where

$$g \le 2a \qquad (1)$$

(a condition imposed by most device packages), the medium filling the cavity is air, and the cavity walls are assumed to be perfectly conducting. Let us investigate the fundamental mode in which E_θ is zero and the fields are independent of θ, and begin by formulating Green's functions for regions I and II which describe the magnetic field in the particular region for the case where the electric field on the surface $r = a$ is $E_z^\delta(a,z)$ given by

$$E_z^\delta(a,z) = -\delta(z - z')\cdot\delta(r - [a \pm 0]), \qquad 0 \le z' \le g$$

where the positive sign is taken for a source in region I and the negative sign for a source in region II. The Green's function for region I can be shown to be

$$H_\theta^{\mathrm{I}}(r,z;z') = -\frac{i}{Zh}\left(\frac{J_1(kr)Y_0(kb) - J_0(kb)Y_1(kr)}{J_0(ka)Y_0(kb) - J_0(kb)Y_0(ka)} + 2\sum_{m=1}^{\infty}\frac{\cos\frac{m\pi z'}{h}\cos\frac{m\pi z}{h}}{q_m}\cdot\frac{K_0(q_mkb)I_1(q_mkr) + K_1(q_mkr)I_0(q_mkb)}{K_0(q_mkb)I_0(q_mka) - K_0(q_mka)I_0(q_mkb)}\right) \qquad (2)$$

where the r and z axes are defined in Fig. 1, and q_m, k, and i are given by

$$q_m = \sqrt{\left(\frac{m\pi}{kh}\right)^2 - 1}, \qquad k = 2\pi/\lambda, \quad i = \sqrt{-1}$$

while Z is the intrinsic impedance of free space. The Green's function for region II, on the other hand, may be shown to be

$$H_\theta^{\mathrm{II}}(r,z;z') = -\frac{i}{Zg}\left(\frac{J_1(kr)}{J_0(ka)} + 2\sum_{m=1}^{\infty}\frac{\cos\frac{m\pi z'}{g}\cos\frac{m\pi z}{g}}{q_m^*}\cdot\frac{I_1(q_m^*kr)}{I_0(q_m^*ka)}\right) \qquad (3)$$

where

$$q_m^* = \sqrt{\left(\frac{m\pi}{kg}\right)^2 - 1}.$$

The magnetic field in each region of the cavity may be found by convolving the appropriate Green's function with the aperture electric field $E_z(a,z)$, and the solution of the problem found by enforcing the continuity of the magnetic field across the aperture, namely

$$H_\theta(a,z) = -\int_0^g E_z(a,z')\cdot H_\theta^{\mathrm{I}}(a,z;z')\,dz' = -\int_0^g E_z(a,z')\cdot H_\theta^{\mathrm{II}}(a,z;z')\,dz', \qquad 0 \le z \le g \tag{4}$$

where $H_\theta(a,z)$ is the magnetic field in the aperture.

Now if $E\,(a,z)$ were known, the resonant frequency of the cavity could be found by solving (4) for any value of z satisfying $0 \le z \le g$. (NB, Hansen [1] chose $z = g$.) Alternatively, we could solve any equation derived from (4). For reasons which will become clearer shortly, let us equate the average value of the magnetic field $H_\theta(a,z)$ in the gap aperture obtained for the two regions, namely

$$\frac{1}{g}\int_0^g H_\theta(a,z)\,dz = -\frac{1}{g}\int_0^g\int_0^g E_z(a,z')H_\theta^{\mathrm{I}}(a,z;z')\,dz'\,dz = -\frac{1}{g}\int_0^g\int_0^g E_z(a,z')H_\theta^{\mathrm{II}}(a,z;z')\,dz'\,dz. \tag{5}$$

Now using (2) and (3) it can be shown that (5) reduces to

$$-\frac{i}{Zg}\cdot\frac{J_1(ka)}{J_0(ka)} = -\frac{i}{Zh}\left(x_0 + \sum_{m=1}^{\infty}\frac{\sin mc}{mc}\cdot x_m \cdot \int_0^g e_z(z')\cos\frac{m\pi z'}{h}\,dz'\right) \tag{6}$$

where

$$x_0 = \frac{J_1(ka)Y_0(kb) - J_0(kb)Y_1(ka)}{J_0(ka)Y_0(kb) - J_0(kb)Y_0(ka)}$$

and

$$x_m = \frac{2}{q_m}\cdot\frac{K_0(q_mkb)I_1(q_mka) + K_1(q_mka)I_0(q_mkb)}{K_0(q_mkb)I_0(q_mka) - K_0(q_mka)I_0(q_mkb)}, \qquad m \ge 1$$

while

$$e_z(z') = \frac{E_z(a,z')}{\int_0^g E_z(a,z)\,dz} \quad \text{and} \quad c = \frac{\pi g}{h}.$$

The advantages of using (5) instead of (4) are now evident. In the first place, the left-hand side (LHS) of (6) consists of only one term and the right-hand side (RHS) of (6) contains only one infinite series. Furthermore, it is known that the form of the RHS of (6) is such that its numerical value is considerably less sensitive to error in $E_z(a,z)$ than is $H_\theta(a,z)$ as calculated by (4). As a result, it is to be expected that (6) would more accurately predict the resonant frequency than would (4) if an approximation was used for $E_z(a,z)$, the method used shortly to solve (6).

It must be remembered, of course, that the exact form of $E_z(a,z)$ could be found by solving an integral equation, a very formidable task indeed. Alternatively, the transcendental equation for the problem could be cast into a variational form and numerical results obtained by using a first-order solution for $E_z(a,z)$. Neither of these approaches are particularly simple analytically nor are the resulting equations, in general, simple to program for rapid numerical solution. On the other hand, (6) is a simple and exact equation which will yield, for the reasons discussed previously, quite accurate results for the resonant frequency if a reasonably accurate approximation is used for $E_z(a,z)$. Now recall that we are principally concerned with narrow-gap situations given by (1) and let us take the following approximation for $e_z(z)$:

$$e_z(z) = -\left(\frac{0.650\,804}{g^{1/3}\cdot\sqrt[3]{g^2 - z^2}} + \frac{0.187\,976}{g^{5/3}}\cdot\sqrt[3]{g^2 - z^2}\right).$$

It has been shown [4], [5] that this approximation is within ± 0.6 percent of the exact result for the limiting case $g \to 0$. Furthermore, it has also been estimated [5] for $g \le a$ that $e_z(z)$ is within ± 2 percent of the actual electric field distribution. The use of this approximation is therefore most appropriate. Using this approximation (6) becomes

$$\frac{J_1(ka)}{J_0(ka)} = \frac{g}{h}\left\{x_0 + \sum_{m=1}^{\infty} x_m \frac{\sin mc}{mc} \cdot \left(C\cdot\frac{J_{1/6}(mc)}{(mc)^{1/6}} + D\cdot\frac{J_{5/6}(mc)}{(mc)^{5/6}}\right)\right\} \tag{7}$$

where $C = 0.876\,644$ and $D = 0.265\,061$. It is this transcendental equation which is solved numerically for the resonant frequency. (The numerical aspects are discussed in the Appendix.) It is interesting that (7), which ought to quite accurately predict the resonant frequency for the case where g is small as a result of the excellent approximation used for $e_z(z)$ in that case, also predicts the correct result when $g = h$ because of the manner in which the continuity condition was applied, namely (5).

IV. Comparison of Theoretical and Experimental Results

In Table I, a comparison is made between the theoretical results for the resonant frequency, f_w, calculated from (7) and the experimental results, f_m, of Uenakada [2]. The percentage error between f_w and f_m, namely e_w, is also shown in Table I together with the percentage error [2], e_u, between f_m and the results computed by Uenakada's formulation. It can be seen that (7) predicts the resonant frequency to better than 1 percent except for cavity 6 where $e_u = 1.45$ percent. In view of the fact that the dimensions of cavities 3 and 6 are similar, it is difficult to rationalize the vast difference in the accuracy of f_w for the two situations, 0.53 percent in one case and 1.45 percent in the other. It seems reasonable, therefore, to suppose that f_m for cavity 6 is slightly in error and that e_w for this cavity is likewise about 0.6 percent.

Note that for $g \le 2a$ (excluding cavity 6) the resonant frequency predicted by (7) is accurate to about 0.7 percent on the average, clearly improving as g/a and g/h decrease. Furthermore, it is interesting to observe that for these cases Uenakada's theoretical results, obtained from a formulation which is both mathematically and computationally more complicated than that presented here, are only accurate to

TABLE I
A COMPARISON OF THE THEORETICAL AND EXPERIMENTAL RESULTS FOR THE RESONANT FREQUENCIES OF A NUMBER OF REENTRANT CYLINDRICAL CAVITIES

Cavity No.	h(mm)	g(mm)	a(mm)	b(mm)	f_m(GHz)	f_w(GHz)	e_w(%)	e_u(%)
1	22.792	7.958	6.004	42.29	2.135	2.1244	0.49	4.17
2	34.826	8.028	5.992	13.8	2.326	2.3086	0.73	2.14
3	31.806	7.984	5.9935	20.99	2.280	2.2678	0.53	4.02
4	28.019	7.999	5.999	29.988	2.2264	2.2164	0.44	4.90
5	31.806	7.980	3.495	20.99	2.394	2.3749	0.79	0.11
6	33.806	10.000	8.405	20.99	2.3027	2.2689	1.45	4.4
7	33.806	10.100	4.206	20.99	2.4018	2.3789	0.95	0.004

Note: f_m is the resonant frequency measured by Uenakada; f_w is that calculated by (7); while e_w and e_u are the percentage errors between f_m and f_w, and f_m and the theoretical results of Uenakada, respectively.

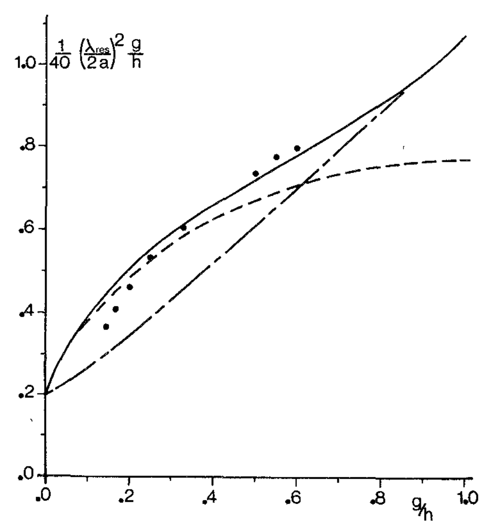

Fig. 2. Plot of $\frac{1}{40}(\lambda_{res}/2a)^2\, g/h$ versus g/h for $b/a = 5$ and $h/a = 2.12$: ———— theoretical results trom (7); —— — —— theoretical results of Rivier and Vergé-Lapisardi [3]; - - - - - - - theoretical results from Fujisawa's formulation [12]; .... experimental results of Rivier and Vergé-Lapisardi [3].

about 3 percent on the average. This is largely attributable to the excellence of the approximation used for $e_z(z)$ and the manner in which the continuity condition was applied. Clearly then, (7) is a most suitable equation from which to calculate the resonant frequencies of reentrant cavities currently being used in the design of solid-state equipment. Furthermore, in view of its accuracy and computational simplicity, (7) represents a significant advance, for such cases, on formulations previously proposed.

In Fig. 2 the parameter $(1/40)(\lambda_{res}/2a)^2(g/h)$ is plotted as a function of g/h (where λ_{res} is the resonant wavelength) as calculated by (7) for the case considered by Rivier and Vergé-Lapisardi [3], namely $b/a = 5$ and $h/a = 2.12$. Also plotted in Fig. 2 are the experimental results of Rivier and Vergé-Lapisardi and the theoretical results calculated by Fujisawa's formulation [12]. From this figure it is clear that while our theoretical results and Rivier and Vergé-Lapisardi's experimental results agree quite well when $0.2 < g/h \leq 0.6$, the agreement is far from satisfactory in the range $0 < g/h \leq 0.2$. It has already been established that theoretical results from (7) for this range of values of g/h, g/h being the parameter which essentially controls the accuracy of (7), are in error by less than about 0.7 percent, and it must therefore be concluded that Rivier and Vergé-Lapisardi's experimental results for $g/h \leq 0.2$ are in error. Note also that results calculated from Fujisawa's formulation which were said by Rivier and Vergé-Lapisardi to be inaccurate are, in fact, in this case, quite accurate for small g/h. Finally, observe from Fig. 2 that the approximately linear portion of the curve, which was the characteristic used by Rivier and Vergé-Lapisardi to evaluate the magnitude of the components of their equivalent circuit, is in fact more extensive than their experimental work indicated.

V. INCORPORATION OF A SEMICONDUCTOR DEVICE

In the theoretical work presented in this paper so far, only the air-gap situation has been considered, although it has been intimated that the analysis may be easily modified to consider the particular case where a semiconductor device is placed in the gap. The situation in the gap region then becomes that shown in Fig. 3 where g, the height of the gap, is dictated by the device size; and d, the radius of the device, is usually less than a, the central post radius. In some situations one may take $a = d$.

Let us consider, briefly, the case of a solid-state oscillator and for the present assume that the cavity is not connected to any external load. If the device admittance, Y_D, is taken to be given by

$$Y_D = \frac{2\pi d}{g} \cdot \frac{\int_0^g H_\theta(d,z)\, dz}{\int_0^g E_z(d,z)\, dz}$$

where $E_z(d,z)$ and $H_\theta(d,z)$ are the electric and magnetic fields, assumed rotationally symmetric, on the device

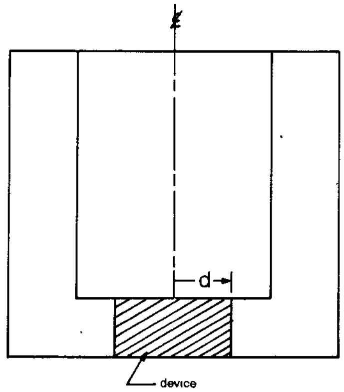

Fig. 3. Cross section of the cavity with a semiconductor device in position.

surface, then it can be shown [5] that the transcendental equation for the resonant frequency is

$$\operatorname{Im}\left(\frac{Zh}{2\pi a} Y_D'\right) = x_0 + \sum_{m=1}^{\infty} x_m \frac{\sin mc}{mc} \cdot \left(C \cdot \frac{J_{1/6}(mc)}{(mc)^{1/6}} + D \cdot \frac{J_{5/6}(mc)}{(mc)^{5/6}}\right) \quad (8)$$

where Y_D' is the device admittance referred to the surface $r = a$. (Whether oscillation would occur in practice is, of course, dependent on the satisfaction of other criteria.) Now if we included consideration of the means by which the cavity was connected to an external load we would obtain yet another equation for that particular situation. However, the important point to appreciate is that if the load is only lightly coupled to the cavity, then the resonant frequency is about the same as, and varies in a similar manner to, that given by (8).

In view of the wide variety of possible coupling systems we do not propose to pursue this avenue further, but rather to conclude this paper with an investigation of the tuning characteristics of an air-gap cavity as predicted by (7), since this provides some qualitative appreciation of the electrical behavior of the cavity.

VI. Tuning Characteristics

In Fig. 4 ka_{res}, the value of ka at resonance, is plotted as a function of h/a for $b/a = 5$ and various values of g/a while Fig. 5 is a plot of ka_{res} versus h/a for $g/a = 1$ and various values of b/a. (NB, the curves have been plotted as a function of h/a since, in practice, the cavity height is frequently used as the means of adjusting the resonant frequency.) From these curves it can be seen that for a given increment in h the change in the resonant frequency is greater for small values of g/a (b/a fixed) than for larger values of g/a and is also greater for smaller values of b/a (g/a fixed) than for larger values of b/a. The advantage one takes of these features depends, of course, on the application in mind. For example, if one was designing a tunable oscillator with the cavity height being the tuning parameter, one might choose a small value of b/a and a small value of g/a in order to obtain a wide frequency variation for a given limited variation in h. On the other hand, if good frequency stability was required from a fixed frequency oscillator, larger values for both b/a and g/a might be chosen. (NB, the maximum value of g/a is determined by the device size.)

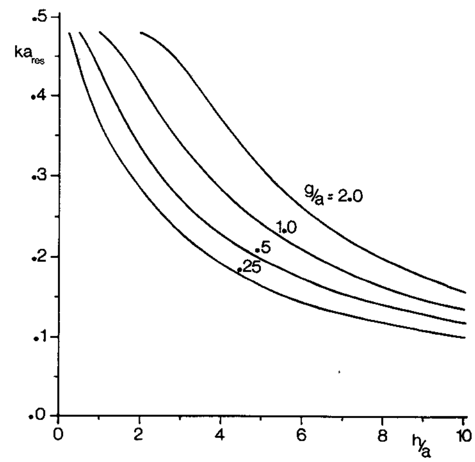

Fig. 4. Variation of ka_{res} as a function of h/a for $b/a = 5$ and various values of g/a.

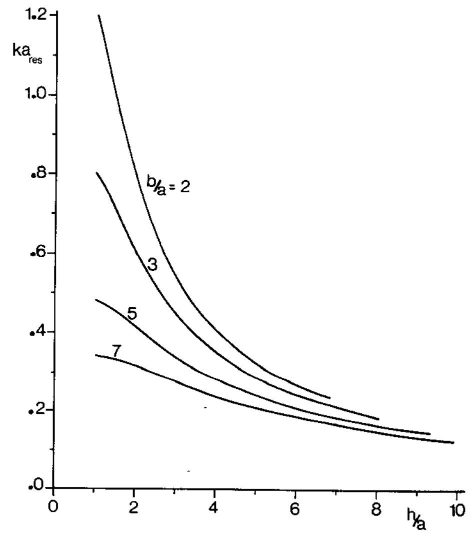

Fig. 5. Variation of ka_{res} as a function of h/a for $g/a = 1$ and various values of b/a.

VII. Conclusion

The reentrant cylindrical cavity has been investigated in this paper and we have seen that over the years a number of different formulations have been published from which the properties of the cavity, and in particular the resonant frequency, may be calculated. This cavity has recently found application in the construction of solid-state oscillators and attention has been drawn to recent theoretical attempts to obtain accurate results for the resonant frequency of cavities suitable for mounting semiconductor devices. In this paper a formulation has been developed, being basically a superior form of Hansen's approach [1], which predicts the fundamental resonant frequency of the cavity to better than 1 percent if $g < 2a$. Furthermore, it has been demonstrated that this new theory is mathematically simpler and numerically more efficient than alternative formulations which are not as accurate.

Appendix

The numerical solution of (7) is usually required either for the situation where a, b, h, and the resonant frequency are prescribed and g is required to be found, or where a, b, h, and g are prescribed and the resonant frequency is required to be found. For both of these cases it is possible to obtain an equation from (7) one side of which is invariant with respect to the unknown quantity, namely (7) for the former case and (7) divided by the term in parenthesis on the RHS of (7) for the latter. The results for the two cases $g/h = 0$ and $g/h = 1$, which are known analytically, may then be used to initiate an iterative solution procedure in which a result for the unknown is obtained from the previous results by a curve-fitting technique. This iteration procedure may be continued until the unknown is evaluated to the required precision. Solution techniques of this kind have been used frequently in the past and, as such, the details will not be further discussed here.

The numerical work involved in (7), or modifications of it, is for the most part very simple, the exception being the evaluation of the sum of the infinite series. For narrow-gap situations it can be shown that

$$\sum_{m=1}^{\infty} x_m \cdot \frac{\sin mc}{mc}\left(C\cdot\frac{J_{1/6}(mc)}{(mc)^{1/6}} + D\cdot\frac{J_{5/6}(mc)}{(mc)^{5/6}}\right)$$
$$\simeq \sum_{m=1}^{M}\left\{\left(x_m + \frac{2kh}{\pi}\cdot\frac{1}{m} + \left(\frac{kh}{\pi}\right)^2\cdot\frac{1}{ka}\cdot\frac{1}{m^2}\right)\cdot\frac{\sin mc}{mc}\right.$$
$$\cdot\left(C\cdot\frac{J_{1/6}(mc)}{(mc)^{1/6}} + D\cdot\frac{J_{5/6}(mc)}{(mc)^{5/6}}\right)$$
$$\left. + \frac{2kh}{\pi}\cdot\beta_m\cdot\left(\frac{\sin mc}{mc}\right)^2\right\}$$
$$-\frac{kh}{\pi c^2}\left(\tfrac{5}{4}(1-\cos[2c]) + \sin^2 c(2\ln[2\sin c]-1)\right)$$
$$-\left(\frac{kh}{\pi}\right)^2\cdot\frac{1}{ka}\cdot\frac{1}{96c^2}\cdot(\pi^4 + [\pi - c]^4 - 2\pi^2[\pi - c]^2)$$
$$+\frac{2kh}{\pi}\cdot\left(0.04757 - 0.03529\cdot\left(1 - \frac{c/2}{\tan c/2}\right)\right)$$
$$+0.01765\cdot\left(\frac{kh}{\pi}\right)^2\cdot\frac{c}{ka}\cdot(\pi - c)$$

where

$$\beta_m = \frac{1}{m(m^2-1)} \quad \text{for} \quad m \geq 2 \text{ and } \beta_1 = 0.$$

This expression is accurate to within ± 0.01 percent if M is chosen to satisfy the relationship

$$M < 1.6\frac{h}{a} \leq M + 1.$$

A detailed derivation of the foregoing result may be found elsewhere [5].

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